MTH 166

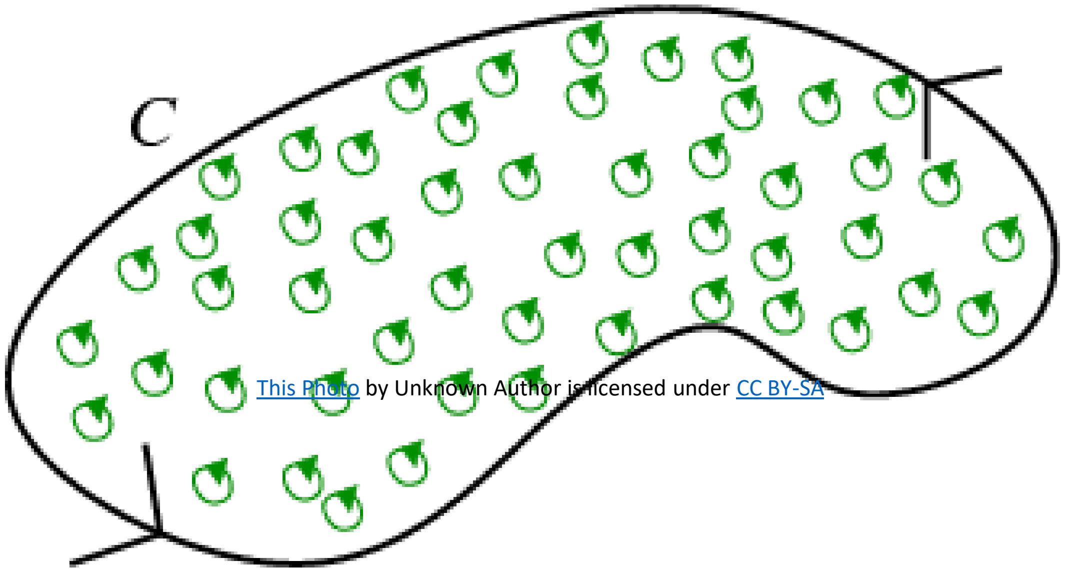
Lecture-31

Greens' Theorem

Statement:

Let C be a piecewise smooth simple closed curve bounding a region R traced in anticlockwise direction. If f and g are two scalar functions which are continuous and have continuous first order partial derivatives on R, then:

$$\oint_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$



<u>Important Results for MCQ Practice:</u>

1. By Greens' theorem: $\oint_C \; f dx + g dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$

2. Greens' theorem is a relationship between double integral and line integral.

3. Greens' theorem is also called as First fundamental theorem of integral vector calculus.

4. Area of region R = $\oint_C \; xdy = -\oint_C \; ydx = \frac{1}{2}\oint_C \; (xdy - ydx)$

5. If $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$, then by Greens' theorem $\oint_C \; f dx + g dy = 0$

Problem: Use Greens' theorem to evaluate: $\oint_C (x+y)dx + x^2dy$, where C is a triangle with the vertices (0,0),(2,0) and (2,4) taken in the order.

Solution: Here the given integral is: $\oint_C (x+y)dx + x^2dy$

Comparing it with: $\oint_C fdx + gdy$

$f = (x+y)$ implies $\frac{\partial f}{\partial y} = 1$, $g = x^2$ implies $\frac{\partial g}{\partial x} = 2x$

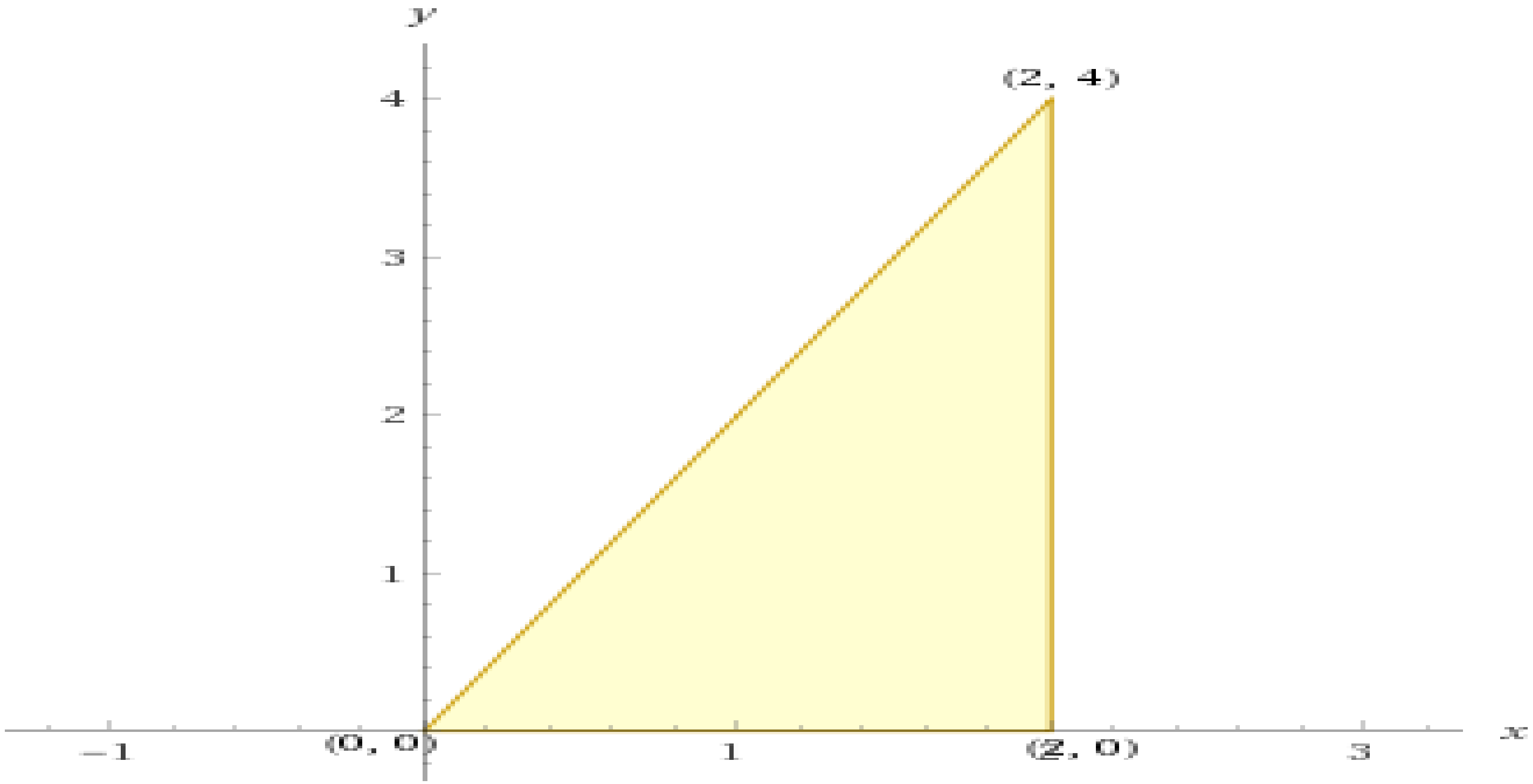
By Greens' theorem:

$$\oint_C fdx + gdy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

$$\oint_C (x+y)dx + x^2dy = \iint_R (2x-1)dxdy$$

Now we are to get the limits of x and y for the evaluation of double integral.

Let us first draw the figure and find limits



In the given figure: $R: \begin{Bmatrix} 0 \le x \le 2 \\ 0 \le y \le 2x \end{Bmatrix}$

Now, Let us evaluate the integral:

$$\oint_C (x+y)dx + x^2 dy = \iint_R (2x-1)\,dxdy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2x} (2x-1)\,dydx$$

$$= \int_{x=0}^{2} (2xy - y)_{y=0}^{y=2x} dx$$

$$= \int_{x=0}^{x=2} (4x^2 - 2x)\,dx$$

$$= \left[4\left(\frac{8}{3}\right) - 4\right]$$

$$= \frac{20}{3}$$ **<u>Answer</u>**

Problem: Use Greens' theorem to evaluate: $\oint_C \; x^3 dy - y^3 dx$, where C is a the circle: $x = 2\cos\theta, y = 2\sin\theta\,, 0 \le \theta \le 2\pi$.

Solution: Here the given integral is: $\oint_C \; x^3 dy - y^3 dx$

Comparing it with: $\oint_C \; f dx + g dy$

$f = -y^3$implies $\frac{\partial f}{\partial y} = -3y^2$, $\;\; g = x^3$ implies $\frac{\partial g}{\partial x} = 3x^2$

By Greens' theorem:

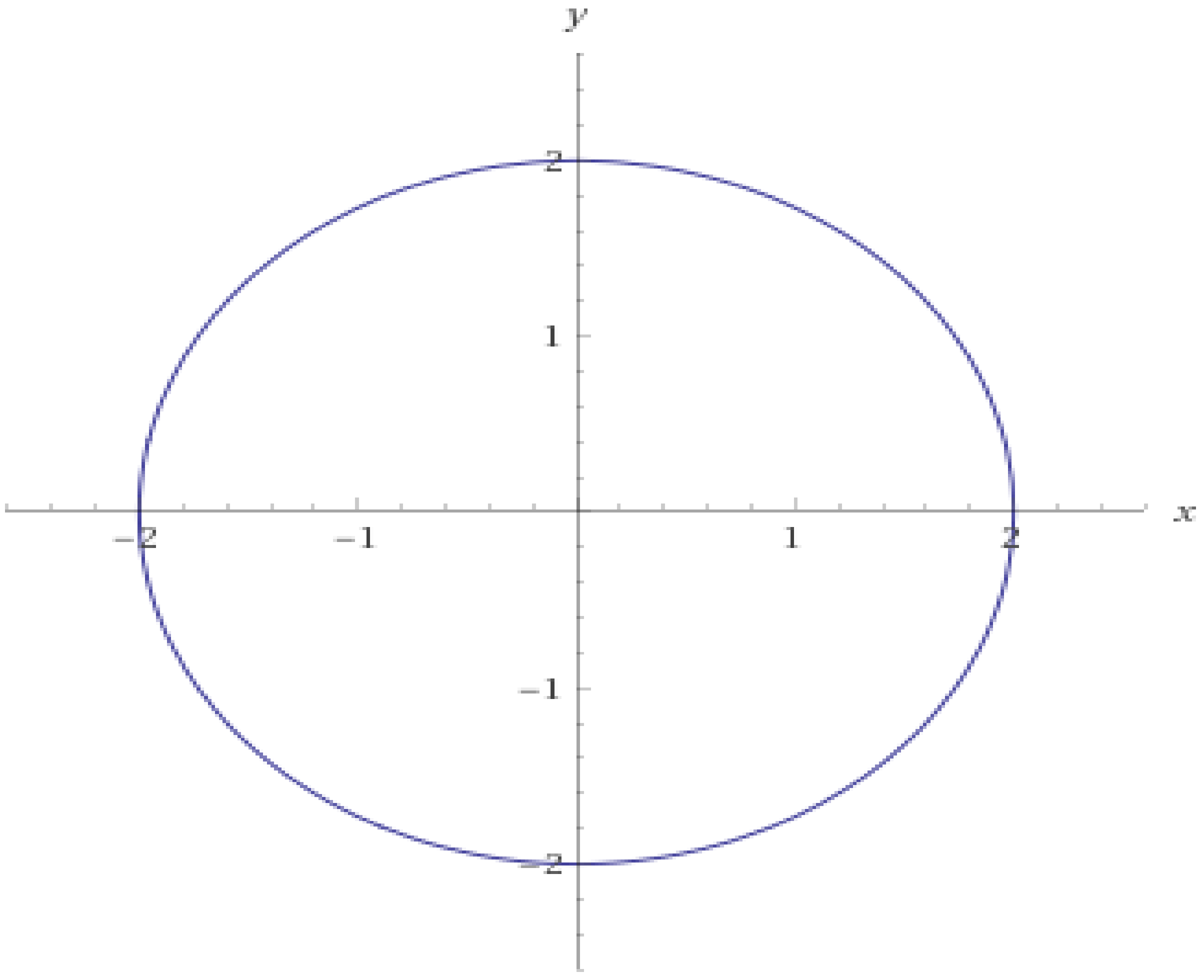
$$\oint_C \; f dx + g dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx dy$$

$$\oint_C \; x^3 dy - y^3 dx = 3 \iint_R (x^2 + y^2) dx dy$$

Here limits of x and y are in polar coordinates:

$x = 2\cos\theta, y = 2\sin\theta\,, 0 \le \theta \le 2\pi$

In the given region R: $0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$



$$\oint_C \; x^3dy - y^3dx = 3\iint_R \; (x^2+y^2)dxdy$$

$$= 3\int_{r=0}^{2}\int_{\theta=0}^{2\pi}(r^2)rdrd\theta$$

$$= 3\int_{r=0}^{2} r^3dr \int_{\theta=0}^{2\pi} d\theta$$

$$= 3\left[\frac{r^4}{4}\right]_{r=0}^{2} [\theta]_{\theta=0}^{2\pi}$$

$$= 3(4)(2\pi)$$

$= 24\pi$ **Answer**

Line Integral Independent of Path of Integration

An integral of the form: $\int_P^Q f(x,y)dx + g(x,y)dy$ is independent of path of integration if and only if: $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

Problem: Show that the integral $\int_P^Q 2xy^2dx + (2x^2y + 1)dy$ is independent of path of integration.

Solution: Compare the given integral with: $\int_P^Q f(x,y)dx + g(x,y)dy$

Here $f = 2xy^2$ and $g = (2x^2y + 1)$

This implies: $\frac{\partial f}{\partial y} = 4xy$ and $\frac{\partial g}{\partial x} = 4xy$

Since, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 4xy$

So, the given integral is independent of path of integration.